

# Precise Time Evolution of Superconductive Phase Qubits

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New procedure on precise analysis of superconducting phase qubits using the concept of Feynman path integral in quantum mechanics and quantum field theory has been introduced. The wave-function and imaginary part of the energy of the pseudo-ground state of the Hamiltonian in Phase Qubits has been obtained from semi-classical approximation and we estimate decay rate, and thus the life time of meta-stable states using the approach of Instantons model. We devote the main efforts to study the evolution of spectrum of Hamiltonian in time after addition of interaction Hamiltonian, in order to obtain the high fidelity quantum gates.

## I. INTRODUCTION

The potential to manipulate information efficiently with quantum mechanics and remarkable promise of quantum computation has led to a search and invention of a significant number of proposals for physical systems that could implement a quantum computer in large size [1–4]. Superconducting circuits using Josephson junctions provide a promising approach towards the construction of a scalable solid-state quantum computer. These devices show the quantum effects in macroscopic scale and it's the most advantage of these elements [5],[6]. They can play the basic building blocks of quantum computers, which are qubits. In addition by manipulating the qubits via external controlled current sources there is possibility to construct the specific quantum gates [7].

A viable quantum computer needs to have a stable and long-live Qubits that make their coherency for long time before the manipulation and operation acts on them.

Thus in order to building the quantum bits and quantum gates with high accuracy and high fidelity we need to have a deep recognition to the exact description physics of the system which in translation to quantum mechanics, it means we should have a precise analysis on evolution of Propagator of the system which is completely time-dependent.

For the basic operation in quantum computation we fundamentally need to study the evolution of  $N$  two-level quantum systems which we can describe their states with the  $N$ -component vector  $|\psi\rangle$ . The evolution of this state can be given by propagator. In the general, the Hamiltonian is uncommutative during the time,  $[H(t), H(t_0)] \neq 0$  and we are bound to use the approximating method such as Dyson series for finding the general form propagator. Unfortunately it's not easy to calculate especially when we need analytical description of system. In the other hand there are various methods to solve the time-independent quantum mechanics problem such as perturbation theory or WKB theory. But in fact WKB is uncontrolled

approximation in general and it is hard to say that the result of this method is accurate or not. Therefore finding the methods that help us to get the more accurate and reliable result is very important and essential.

In this paper we claim that functional formalism of quantum mechanics and Feynman path integral [8],[9] give us the more accurate answer about estimating the ground states of energy and describing the meta-stable states wave functions and decay rates of states in superconducting phase qubits. Admittedly the formalism of path integral has been built completely time dependent and evolutionary process of the system will be tracked more conveniently. This is the biggest treasure that lies down under this formalism.

At first section we review on structure of phase potential and we use from the Instanton model for finding the most properties of ground states of energy up to accuracy of  $O(\hbar)$ , after that we present the time-dependent propagator of the quantum system and this achievement leads us to find the representation of the Hamiltonian across the time while the application of external manipulation, for the application of building the quantum gates.

## II. SUPERCONDUCTING PHASE QUBIT

Single Josephson junction phase Qubits consist of one Josephson junction which uses the quantum tunneling effect to produce the continuous current in the existence of external current source,  $I_e$ .

The Hamiltonian of system can be written as

$$H_{dc} = -E_C \frac{\partial^2}{\partial \delta^2} + E_J \cos \delta + \frac{\hbar}{2e} I_e \delta \quad (1)$$

Where  $\delta$  presents the phase of Josephson junction and  $E_J = \hbar/2e I_e$ . In order to manipulate the system we need to evolve the system by time-dependent current, this manipulation introduces the Hamiltonian of interaction which can be given by

$$H_{\mu\nu} = \frac{\Phi_0}{2\pi} I_{\mu\nu} \delta = \frac{\Phi_0}{2\pi} I(t) \cos(\omega t + \phi) \delta \quad (2)$$

Here  $\Phi_0 = \frac{h}{2e}$  is quantum of flux, Thus the total Hamiltonian

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of system yields

$$H(t) = H_0 + V(t) = H_0 + \frac{I_0 \Phi_0}{2\pi} I(t) \cos(\omega t + \phi) \quad (3)$$

### III. INSTANTON MODEL

In this section we study the metastable states in tilted-washboard potential of Josephson-junction phase qubit. We use from path integral approach for our study. If we consider the particle with unite mass which is under the influence of the one-dimentional potential  $V(x)$ , then following the Euclidean form of the Feynman path integral we can describe the evolution of the particle with

$$\langle x_f | e^{-\frac{HT}{\hbar}} | x_i \rangle = N \int [dx] e^{-\frac{S}{\hbar}} \quad (4)$$

Here  $|x_f\rangle$  and  $|x_i\rangle$  are the eigenvalue of the space and the  $N$  refer to normalization factor,  $H$  represent the Hamiltonian of the system which can be depends on time and  $T$  shows the time interval of the evolution.

the symbol of  $[x]$  denotes the integration over the all functions  $x(t)$  that obey from the boundarycondition  $x(-\frac{T}{2}) = x_i, x(\frac{T}{2}) = x_f$  If  $\bar{x}$  be any functions which obeys the boundary condition then we can write  $x(t) = \bar{x}(t) + \sum_n c_n x_n(t)$  where the set  $x_n$  build the complete set,  $\int_{-\frac{T}{2}}^{\frac{T}{2}} x_n(t) x_m(t) = \delta_{mn}$  and  $x_n(\pm\frac{T}{2}) = 0$ . By these condition we can rewrite the mesure of the integral by

Calculation in order of  $\hbar$  and using the semi classical approximation lets us to write the evolution of the systme by

$$\begin{aligned} \langle x_f | e^{-\frac{HT}{\hbar}} | x_i \rangle &= N e^{-\frac{S(\bar{x})}{\hbar}} \prod_n \lambda_n^{-\frac{1}{2}} [1 + O(\hbar)] \\ &= N e^{-\frac{S(\bar{x})}{\hbar}} [\det(-\partial_t^2 + V''(\bar{x}))]^{-\frac{1}{2}} [1 + O(\hbar)] \end{aligned} \quad (5)$$

If we define  $\omega^2$  to be  $V''(0)$  then the standard calculation shows that for large  $T$

$$N [\det(-\partial_t^2 + \omega^2)]^{-\frac{1}{2}} = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \quad (6)$$

Now if we consider the desire potential shape as Fig.III

Here we still asumme that  $x_i = x_f = 0$ . If we look at the inversed potential ( $-V(x)$ ) we can see despite the previous case, here we can have the nontrivial answer. the particle can roll itself down from the top of the hill of potential at  $x = 0$  and goes untill reaches to the turning point. as we interested to limit the time to infnty in the final calculation, then we hold the answers that acure in long time. this situation is so called “the Bounce”. In this situation Energy of particle is zero. Action in this situation give by

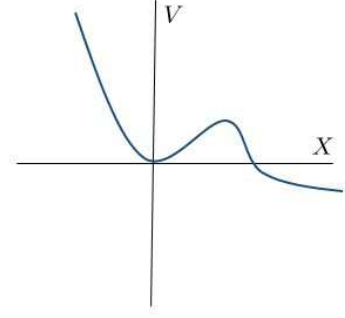


FIG. 1. Special case of potential which provides the bounce motion, it has local minimum, but not the absolute one.

$$B = \int_{-\infty}^{\infty} dt \left(\frac{dx}{dt}\right)^2 = \int_0^{\sigma} dx [2V(x)]^{\frac{1}{2}} \quad (7)$$

where  $x = \sigma$  is a place where poential iz zero,  $V = 0$ . If we define the center of the bounce by the place where we have  $\frac{dx}{dt} = 0$  there, we can see this point is invarinat under time translation. For large time,  $T$ , we can put  $n$  seprated points like that that have enough space from here and each of points plays the role of one sigle Bounce motion. If we show the center of this points by  $\frac{T}{2} > t_1 > t_2 \dots > t_n > -\frac{T}{2}$  In path integral we should notice that the value of action chanches from  $S$  to  $nS$  and as this  $n$  points are located in the far distance from each other we can write the determinant as the product of determiniat of many single Bounce motion. in this way we obtain

$$\left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\omega\frac{T}{2}} K^n \quad (8)$$

Where  $K$  is a factor that we will discuss on it later. Also the integration over time yeilds

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\frac{T}{2}}^{t_1} dt_2 \dots \int_{-\frac{T}{2}}^{t_{n-1}} dt_n = \frac{T^n}{n!} \quad (9)$$

finally we will have

$$\sum_{n=0}^{\infty} \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{\omega T}{2}} \frac{(K e^{-\frac{B}{\hbar}} T)^n}{n!} = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{[-\frac{\omega T}{2} + K e^{-\frac{B}{\hbar}} T]} \quad (10)$$

From Eq.4 we can see for example that the correspond value of ground state of enregy can be given by

$$E_0 = (\hbar\omega - \hbar K e^{-\frac{B}{\hbar}}) [1 + O(\hbar)] \quad (11)$$

We can see that one of the eigenvalue is zero and the corresponding eigenfunction that provide this eigenvalue can be easily given by  $x_1 = B^{-\frac{1}{2}} \frac{dx}{dt}$ , therefore we need to omitte this trubling zero eigenvlue. we can use many ways to solve this problem and by the standardr way we

can calculate prime determinant which the zero eigenvalue has been omitted and we need to add coefficient to the  $K$  like as  $(\frac{B}{2\pi\hbar})^{\frac{1}{2}}$  [10],[11].

Now if we review on our solution way with more details we will find that as in one place the  $\frac{d\bar{x}}{dt}$  become zero there for  $x$  has node and thus it cannot be the lowest eigenvalue of energy. It means that this system has the negative eigenvalue and the spectrum of our systems is the special case and we have unstable state here and the unitarity of this location from the Hilbert space can be in doubt.

The key to point for solving this problem is that we need to add the coefficient one-half to our calculation and the reliable result yields [10],[11],[12]

$$\text{Im}[N \int [dx] e^{-\frac{S}{\hbar}}] = \frac{1}{2} N e^{-\frac{B}{\hbar}} \left(\frac{B}{2\pi\hbar}\right)^{\frac{1}{2}} T |\det' [-\partial_t^2 + V''(\bar{x})]|^{-\frac{1}{2}} \quad (12)$$

and by comparing the result with the definition of  $K$  we will find that

$$\text{Im}K = \frac{1}{2} \left(\frac{B}{2\pi\hbar}\right)^{\frac{1}{2}} \left| \frac{\det' [-\partial_t^2 + V''(\bar{x})]}{\det [-\partial_t^2 + \omega^2]} \right|^{-\frac{1}{2}} \quad (13)$$

In the stable situation, when the height of barrier penetration goes to infinity the solution of Schrödinger equation, corresponding to the ground state energy  $E_0$  behaves as

$$\psi_0(t) \sim e^{-\frac{iE_0 t}{\hbar}} \quad (14)$$

But for the case that we have not absolute minimum,  $E_0$  becomes imaginary. Therefore for long times we have

$$|\psi_0(t)| \sim e^{-\frac{\text{Im}E_0 t}{\hbar}} \quad (15)$$

It clearly shows that the amplitude and therefore the probability of state decays. The parameter  $|\frac{\hbar}{\text{Im}E_0}|$  is the lifetime of a now metastable state with wave function  $\psi(t)$ . Let us to point out that the decay of state receives contributions from the continuation of all excited states. However, one expects, for intuitive reasons, that when the real part of the energy increases the corresponding contribution decreased faster with time, a property that can, indeed, be verified in examples. Thus, for large times, only the component corresponding to the pseudo-ground state survives. by considering the one-half calculation we have

$$\begin{aligned} \Gamma &= -2\text{Im}E_0/\hbar \\ &= \left(\frac{B}{2\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{B}{\hbar}} \left| \frac{\det' [-\partial_t^2 + V''(\bar{x})]}{\det [-\partial_t^2 + \omega^2]} \right|^{-\frac{1}{2}} [1 + O(\hbar)] \end{aligned} \quad (16)$$

#### IV. ESTIMATION OF THE COEFFICIENT

Here we have similar situation to unstable states and bounces, therefore we follow the mentioned solving way that we discussed in previous sections

In order to finding the classical path we should inverse the potential. If we call the turning point by  $\sigma$ , as is clear in fig 12, then  $\sigma$  is the zero of  $V(x) = \alpha \cos x + \beta x + \epsilon(x)$ . The analytical solution of this equation obviously is not clear at first sight, specially the correction error,  $\epsilon(x)$ , has no simple formula. Thus it's better to solve it with soft wares, depend on our parameter. By knowing the turning point then estimating the action,  $S_0$  is easy. as usual

$$S_0 = \int_{\delta_i}^{\delta_f} d\delta \sqrt{\frac{2V(\delta)}{m}} \quad (17)$$

From our Hamiltonian it is clear that  $m = \frac{\hbar^2}{2E_C}$  and  $V(\delta) = E_J \cos \delta + \frac{\hbar}{2e} I_e \delta + \epsilon(\delta)$ . Thus

$$S_0 = \int_0^\sigma d\delta \sqrt{\frac{4E_C}{\hbar^2}} \sqrt{E_J \cos \delta + \frac{\hbar}{2e} I_e \delta + \epsilon(\delta) + c_0} \quad (18)$$

Where  $c_0$  is constant that appear from changing the coordinate in order to the hill point of potential locate at zero point of coordinate. Value of this integral can be calculate easily by soft wares. Now we try to find the classical path. For simplicity and consistency we previous formula in previous subsection we change the variable of motion  $x$  or  $q$  instead of  $\delta$ . From equation of motion we have

$$\frac{1}{2} m \dot{x}_c^2 = V(x_c) + E_0 \quad (19)$$

Thus the classical path obey from this relation

$$\begin{aligned} t &= t_1 + \sqrt{\frac{2}{m}} \int_0^{x'} dx_c \sqrt{V(x_c) + E_0} \\ &= \int_0^{x'} dx_c \frac{\sqrt{\frac{\hbar^2}{4E_C}}}{\sqrt{E_J \cos x_c + \frac{\hbar}{2e} I_e x_c + \epsilon(x_c) + E_0}} \end{aligned} \quad (20)$$

The  $E_0$  is the constant of motion in must be selected which in  $x \rightarrow 0$ ,  $t \rightarrow -\infty$  and vice-versa. As we know the zero Eigenfunction of  $[-\partial_t^2 + V''(x_c)]$  is

$$x_1 = S_0^{-\frac{1}{2}} \frac{dx_c}{dt} \quad (21)$$

For the next estimation we strongly to know the behavior of  $x_1$  respect to time. from previous equation we have function  $t(x_c)$  and what we need is the inverse of this function  $g = f^{-1} = x_c(t)$ . Finding the analytical form for this function is complicated and it's better to solve it numerically in exact case that we need and with desire parameters.

And for example, for the potential  $V(x) = \frac{1}{2}x^2 + \frac{1}{2}gx^4$  the  $x_c(t)$  has the form  $x_c(t) = g(t) \sim \frac{1}{\cosh(t-t_0)}$ , Fig 13. Hence we expect that  $x_1$  behave exponentially when time goes to infinity.

$$x_1 = S_0^{-\frac{1}{2}} \frac{dx_c}{dt} \rightarrow A e^{-|t|}, t \rightarrow \pm\infty \quad (22)$$

We consider estimating the quantities  $S_0$  and  $x_c(t)$  let us to estimate the  $A$  factor which is constant and fundamentally is function of just  $I_C$  and  $I_e$  and capacitance and cross section area of Josephson junction.

It is easy to show that in genral [12]

$$\frac{\det[-\partial_t^2 + U''(x_c)]}{\det[-\partial_t^2 + \omega^2]} = \frac{1}{2A^2} \quad (23)$$

And

$$K = \left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}} \sqrt{\frac{1}{2A^2}} \quad (24)$$

Thus finally we will have

$$\Gamma = \hbar \left(\frac{S_0}{2\pi\hbar}\right)^{\frac{1}{2}} \sqrt{\frac{1}{2A^2}} e^{-\frac{S_0}{\hbar}} \quad (25)$$

## V. PROPERTIES OF PROPAGATOR

Propagator palys the fundematal rules for undersatnding the evolution of the system and the whole properties of the system basicly can obtain from the Propagator and here we want find the element of the Hamiltonian of our sytem which are dependent on time. As we know

$$\langle x_f | U(t_f, t_i) | x_i \rangle = U(x_f, t_f; x_i, t_i) = \int_{x(t_f)}^{x(t_i)} D[x(t)] e^{\frac{i}{\hbar} S[x(t)]} \quad (26)$$

It is clear that the Path Integral are invariant under the transformation of  $x(t) \rightarrow x(t) + y(t)$ ,  $y(t_i) = y(t_f)$  Which yields to Equation which is so called Schwinger-Dyson equation:

$$\int_{x(t_i)=x_i}^{x(t_f)=x_f} [Dx(t)] \frac{\delta}{\delta x(t)} e^{\frac{i}{\hbar} S[x(t)]} = 0 \quad (27)$$

Which is equal to In the other hand as we know

$$S[x(t)] = \int_{t_i}^{t_f} L(x(t), \dot{x}(t); t) dt \quad (28)$$

combining this equation with Schwinger-Dyson equation we found that the Identity

$$\int_{x(t_i)=x_i}^{x(t_f)=x_f} [Dx(t)] \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)} \right) e^{\frac{i}{\hbar} S[x(t)]} = 0 \quad (29)$$

This Identity is some aspect of the Ehrenfest Thoerem.

Now if looking for the variation od Action with conditions which  $\delta x(t_i) = 0$ ,  $\delta x(t_f) \neq 0$ , then we have

$$\begin{aligned} \delta S[x(t)] &= \int_{t_i}^{t_f} dt \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)} \right) \delta x(t) + \frac{\partial L}{\partial \dot{x}(t)} \delta x(t_f) \quad (30) \\ &= \int_{t_i}^{t_f} dt \left( \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}(t)} \right) \delta x(t) + p(t_f) \delta x(t_f) \quad (31) \end{aligned}$$

If we define  $p(t_f) = p_f$  then

$$p_f = \frac{\partial S[x_c]}{\partial x_f} \quad (32)$$

Now the variation of the propagator means

$$\delta U(x_f, t_f; x_i, t_i) = \delta x(t_f) \frac{\partial}{\partial x_f} U(x_f, t_f; x_i, t_i) \quad (33)$$

Therefore

$$\frac{\partial}{\partial x_f} U(x_f, t_f; x_i, t_i) = \frac{i}{\hbar} \int_{x(t_i)=x_i}^{x(t_f)=x_f} D[x(t)] p(t_f) e^{\frac{i}{\hbar} S[x(t)]} \quad (34)$$

The other property that we need to know is the variation of the action with respect to time. As we know

$$L(x_f, \dot{x}_f) = \frac{d}{dt} S[x(t)] = \frac{\partial S}{\partial t_f} + \frac{\partial S[x(t)]}{\partial x_f} \frac{dx_f}{dt} = \frac{\partial S}{\partial t_f} + p_f \dot{x}_f \quad (35)$$

Thus

$$\frac{\partial S[x_c]}{\partial t_f} = L(x_f, \dot{x}_f) - p_f \dot{x}_f = -H(x_f, p_f) \quad (36)$$

Now we can see that

$$i\hbar \frac{\partial}{\partial t_f} \int_{x(t_i)=x_i}^{x(t_f)=x_f} [Dx(t)] e^{\frac{i}{\hbar} S[x(t)]} = \int_{x(t_i)=x_i}^{x(t_f)=x_f} [Dx(t)] H(x_f, p_f) e^{\frac{i}{\hbar} S[x(t)]} \quad (37)$$

Therefore the Propagator is the kernel or The green function of the Schroodinger equation

$$[i\hbar \frac{\partial}{\partial t_f} - H(x_f, p_f)] U(x_f, t_f; x_i, t_i) = 0 \quad (38)$$

## VI. EVOLUTION OF TIME-DEPENDENT HAMILTONIAN

Discution when the systemn is open ...

As we works in the Semi-classica regime we develope our result end evolution with this approximation. If we consider the Action and change it's variable to

$$x(\tau) := y(\tau) + \bar{x}(\tau) \quad (39)$$

$$\begin{aligned} \int L dt &= S(x(\tau)) = S(y(\tau) + \bar{x}(\tau)) \\ &= S(\bar{x}(\tau)) + \frac{\delta S}{\delta x} |_{\bar{x}} y(\tau) + \frac{1}{2} \frac{\delta^2 S}{\delta x^2} |_{\bar{x}} y(\tau)^2 \end{aligned} \quad (40)$$

Then by semi-classical approximation we have

$$S(\bar{x} + y) \simeq S(\bar{x}) + \frac{1}{2}\delta^2 Sy^2 \quad (41)$$

the most advantage of the Path Integral for our works lie under this property that the path integral is completely time-dependent formalism and it's not important does the Hamiltonian is time dependent or not and it's the vital property which we need, in the other hand if we work with canonical formalism there is no clear connection between the time-dependent perturbation and time-independent one.

$$\begin{aligned} U(x_f, t_f; x_i, t_i) &= \int_{x(t_f)}^{x(t_i)} D[x(t)] e^{\frac{i}{\hbar} S[x(t)]} \\ &= \int_{x(t_f)}^{x(t_i)} D[y(t)] e^{\frac{i}{\hbar} [S(\bar{x}) + \frac{1}{2}\delta^2 Sy^2]} \end{aligned} \quad (42)$$

Therefore we found the most important lemma that helps us to develop the calculation :

$$U(x_f, t_f; x_i, t_i) = e^{\frac{iS(x_f, t_f; x_i, t_i)}{\hbar}} U(0, t_f; 0, t_i) \quad (43)$$

which the  $U(0, t_f; 0, t_i)$  is the propagator of the system that has the Hamiltonian  $H = H(t)$  for the special case that the  $x_f = x_i = 0$ .

by this relation we can obtain the general propagator that has the initial and the final positions are arbitrary.

## VII. ELEMENT OF THE HAMILTONIAN

In principle we can obtain the path integral formalism by accepting the two fundamental properties for propagator. the first one, we consider that the propagator has the Markovian behavior.

We consider a bounded operator in Hilbert space,  $U(t, t')$ ,  $t \geq t'$ , which describes the evolution from time  $t'$  to time  $t$  and satisfies a Markov property in time [19]

$$U(t, t'')U(t'', t') = U(t, t') \quad \text{for } t \geq t'' \geq t' \quad (44)$$

Also we consider  $U(t', t') = \mathbf{1}$ . moreover, we assume that  $U(t, t')$  is differentiable with a continuous derivative. We set

$$\frac{\partial U(t, t')}{\partial t} \Big|_{t=t'} = \frac{H(t)}{i\hbar} \quad (45)$$

here  $\hbar$  is a real parameter and as we know later it becomes Planck's constant. With these two fundamental properties we can obtain an interesting result. By differentiating the Eq.44 with respect to  $t$  and taking  $t'' = t$  we find

$$i\hbar \frac{\partial U}{\partial t}(t, t') = H(t)U(t, t') \quad (46)$$

Now we use from this Identity to estimate the Hamiltonian

$$\langle y | \frac{\partial U(t, t')}{\partial t} \Big|_{t=t'} | x \rangle = \frac{1}{i\hbar} \langle y | H(t) | x \rangle \quad (47)$$

as

$$\langle y | \frac{\partial U(t, t')}{\partial t} \Big|_{t=t'} | x \rangle = \frac{\partial}{\partial t} \langle y | U(t, t') | x \rangle \Big|_{t=t'} \quad (48)$$

Thus

$$\begin{aligned} \langle y | H(t) | x \rangle &= i\hbar \left\{ \left[ \frac{\partial}{\partial t_f} e^{i \frac{S(x_f, t_f; x_i, t_i)}{\hbar}} U(0, t_f; 0, t_i) \right] \Big|_{t_i=t_f} \right. \\ &\quad \left. + \left[ e^{\frac{S(x_f, t_f; x_i, t_i)}{\hbar}} \frac{\partial}{\partial t_f} U(0, t_f; 0, t_i) \right] \Big|_{t_i=t_f} \right\} \end{aligned} \quad (49)$$

Hence we find that

$$\begin{aligned} \langle y | H(t) | x \rangle &= i\hbar \left\{ -\frac{i}{\hbar} H(x_f, p_f) \right. \\ &\quad \left. + \left[ \frac{\partial}{\partial t_f} U(0, t_f; 0, t_i) \right] \Big|_{t_i=t_f} \right\} \end{aligned} \quad (50)$$

here for obtaining the result we need to know the  $\frac{\partial}{\partial t_i} U(0, t_f; 0, t_i)$  that by estimating for our system we can easily derive from it. But as we estimated, the propagator for  $x_f = x_i = 0$  is

$$U(0, t_f; 0, t_i) = \left( \frac{\omega}{\pi\hbar} \right)^{\frac{1}{2}} e^{-\frac{i\omega(t_f - t_i)}{2}} e^{-\Gamma(t_f - t_i)} \quad (51)$$

Where  $\Gamma$  can be given by

$$\Gamma = \hbar |K| e^{-\frac{S_0}{\hbar}} \quad (52)$$

Thus

$$\frac{\partial}{\partial t_f} U(0, t_f; 0, t_i) \Big|_{t_i=t_f} = \left( \frac{\omega}{\pi\hbar} \right)^{\frac{1}{2}} \left[ -\frac{i\omega}{2} - \Gamma \right] \quad (53)$$

In our case as the second derivative of time dependent potential,  $V''$ , is independent of time, this estimation is like a time-independent mode.

$$\begin{aligned} \langle y | H(t) | x \rangle &= i\hbar \left\{ -\frac{i}{\hbar} H(x_f, p_f) \right. \\ &\quad \left. + \left[ \left( \frac{\omega}{\pi\hbar} \right)^{\frac{1}{2}} \left[ -\frac{i\omega}{2} - \Gamma \right] \right] \right\} \end{aligned} \quad (54)$$

now if the  $|n\rangle$  and  $|m\rangle$  be the two metastable states of the system that are time-dependent fundamentally, then

$$\begin{aligned} \langle m | H(t) | n \rangle &= \int_x \int_y dx dy \langle m | x \rangle \langle x | H(t) | y \rangle \langle y | n \rangle \\ &= \int_x \int_y m^*(x) \langle x | H(t) | y \rangle n(y) \end{aligned} \quad (55)$$

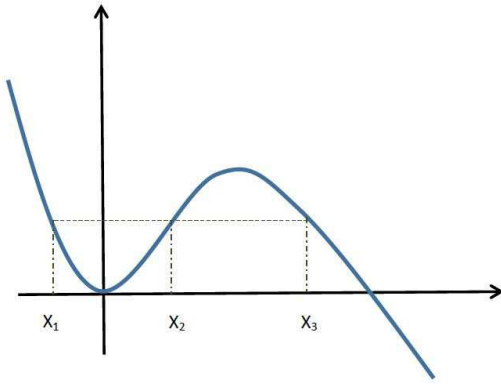
Or

FIG. 2. Tilted Washboard potential.

$$\begin{aligned}
\langle m|H(t)|n\rangle &= \int_x \int_y dx dy \langle m|x\rangle \langle x|H(t)|y\rangle \langle y|n\rangle \quad (56) \\
&= \int_x \int_y dx dy \psi_m^*(x) i\hbar \left\{ -\frac{i}{\hbar} H(x_f, p_f) \right. \\
&\quad \left. + \left[ \left( \frac{\omega}{\pi\hbar} \right)^{\frac{1}{2}} \left[ -\frac{i\omega}{2} - \Gamma \right] \right\} \psi_n(y)
\end{aligned}$$

## Appendix A WAVE FUNCTIONS

As we saw in previous section, estimating the element of the Hamiltonian requires the eigenfunction of the energy and we need the corresponding wave functions in the representation of space. Here we want to obtain the approximated form of them and here we use the semi-classical approximation or in the other word we use the WKB approximation and expand wave function by order of  $\hbar$ .



In the WKB approximation regime we use the the Patch function as a auxiliary function to connecting the coefficient if the wavefunction in the two side of the re-turning piont. Therefore near the turning point the wavefunction is near to solution of the diffrential equation which their answer given by airy function

$$\psi_p = aAi(\alpha x) + bBi(\alpha x) \quad (57)$$

By defining

$$\theta := \frac{1}{\hbar} \int_{x_1}^{x_2} p(x') dx', \quad \gamma := \int_{x_2}^{x_3} |p(x)| dx \quad (58)$$

By comparint the coefficient and by using the patch function near each point we find that

$$\psi(x) \simeq \begin{cases} \frac{D}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar} \int_{x_1}^{x_1} |p(x')| dx'} & x < x_1 \\ -\frac{2D}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar} \int_x^{x_2} p(x') dx' - \theta - \frac{\pi}{4}\right] & x_1 < x < x_2 \\ \frac{D}{\sqrt{|p(x)|}} \left[ 2 \cos \theta e^{\frac{1}{\hbar} \int_{x_2}^x |p(x')| dx'} + \sin \theta e^{-\frac{1}{\hbar} \int_{x_2}^x |p(x')| dx'} \right] & x_2 < x < x_3 \\ \frac{1}{\sqrt{p(x)}} \left[ \frac{D \sin \theta}{e^{\gamma} e^{-i\frac{\pi}{4}}} e^{\frac{1}{\hbar} \int_x^{x_3} p(x') dx'} \right] & x > x_3 \end{cases} \quad (59)$$

As the amount of enrgy for metastable states has imaginary part, therefore the amplitued of wevefuntion decay gradually and the state disappear after long time.

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